PERSISTENT HOMOLOGY OF MORSE DECOMPOSITIONS IN COMBINATORIAL DYNAMICS

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Abstract. We investigate combinatorial dynamical systems on simplicial complexes considered as *finite topological spaces*. Such systems arise in a natural way from sampling dynamics and may be used to reconstruct some features of the dynamics directly from the sample. We study the homological persistence of *Morse decompositions* of such systems as a tool for validating the reconstruction. Our approach may be viewed as a step toward applying the classical persistence theory to data collected from a dynamical system. We present experimental results on two numerical examples.

1. Introduction

The aim of this research is to provide a tool for studying the topology of *Morse decompositions* of sampled dynamics, that is dynamics known only from a sample. Morse decomposition of the phase space of a dynamical system consists of a finite collection of isolated invariant sets, called *Morse sets*, such that the dynamics outside the Morse sets is gradient-like. This fundamental concept introduced in 1978 by Conley [11] generalizes classical Morse theory to non-gradient dynamics. It has become an important tool in the study of the asymptotic behavior of flows, semi-flows and multivalued flows (see [7, 12, 31] and the references therein). Morse decompositions and the associated Conley-Morse graphs [3, 8] provide a global descriptor of dynamics. This makes them an excellent object for studying the dynamics of concrete systems. In particular, they have been recently applied in such areas as mathematical epidemiology [20], mathematical ecology [3, 8, 19] or visualization [32, 33].

Unlike the case of theoretical studies, the methods of classical mathematics do not suffice in most problems concerning concrete dynamics. This is either because there is no analytic solution to the differential equation describing the system or, even worse, the respective equation is only vaguely known or not known at all. In the first case the dynamics is usually studied by numerical experiments. In some cases this may suffice to make mathematically rigorous claims about the system [24]. In the latter case one can still get some insight into the dynamics by collecting data from physical experiments or observations, for instance as a time series [1, 19, 25]. In both cases the study is based on a finite and not precise sample, typically in the form of a data set. The inaccuracy in the data may be caused by

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noise, experimental error, or numerical error. Consequently, it may distort the information gathered from the data, raising the question whether the information is trustworthy. One of possible remedies is to study the stability of the information with respect to perturbation of the data. This approach to Morse decompositions constructed from samples is investigated in [32] in the setting of piecewise constant vector fields on triangulated manifold surfaces. The outcome of the algorithm proposed in [32] is the Morse merge tree which encodes the zero-dimensional persistent homology under perturbations of individual Morse sets in the Morse decomposition. Recall that persistent homology [15, 10] is the main tool of topological data analysis [9], facilitating the investigation of homology in the presence of noise or errors of any source. Thus, it is a natural tool to study the topology of Morse decompositions in dynamical systems known only from numerical or experimental samples.

In this paper we study general persistence of Morse decompositions in combinatorial dynamics, not necessarily related to perturbations. To this end, we define a homological persistence of the Morse decomposition over a sequence of combinatorial dynamics. By combinatorial dynamics we mean a multivalued map acting on a simplicial complex treated as a finite topological space. This general setting may be applied either to a finite sample of the action of a map on a subspace of $\mathbb{R}^d$ [5, 14] or to a combinatorial vector field [18] and its generalization multivector field [27]. The persistence is obtained by linking the homology of topologies induced by Morse decompositions and Alexandrov topology in the sequence of combinatorial dynamical systems connected with continuous maps in zigzag order.

On the theoretical level, the results presented in this paper may be generalized to arbitrary finite T_0 topological spaces. From the viewpoint of applications, the finite topological space may be a collection of cells of a simplicial, cubical, or general cellular complex approximating a cloud of sampled points. The multivalued map may be constructed either from the action of a given map on the set of a sample points or from the available vectors of a sampled vector field. The framework for persistence of Morse decompositions in the combinatorial setting developed in this paper is general and may be applied to many different problems.

The language of finite topological spaces (see Section 2.1) enables us to emphasize differences between the classical and combinatorial dynamics. These differences matter when the available data set is sparse and is difficult to be enriched. In particular, in the classical setting the phase space has Hausdorff topology (T_2 topology) and the Morse sets are compact. Hence, Morse sets are isolated since they are always disjoint. To achieve such isolation in sampled dynamics, one needs data not only in the Morse sets but also between the Morse sets. This may be a problem if the available data set is sparse and cannot be enhanced. Fortunately, the finite topological spaces in general are not T_2. In such a space every set is compact but compactness does not imply closedness. Consequently, Morse sets need not be closed and may be adjacent to one another. By allowing adjacent Morse sets we can detect finer Morse decompositions. We still can disconnect them by modifying slightly the topology of the space without changing the topology of the Morse sets.

The organization of the paper is as follows. In Section 2 we recall preliminary material and notation needed in the paper. In Section 3 we introduce the concept of a combinatorial dynamical system, define solutions and invariant sets of a combinatorial dynamical system and present two methods for constructing combinatorial dynamical systems from data. In Section 4 we define the concepts of isolating neighborhood, isolated invariant set and Morse decomposition of a combinatorial dynamical system. In Section 5 we define homological persistence of Morse decompositions in the setting of combinatorial dynamical systems. In Section 6 we discuss computational aspects of the theory and provide a geometric interpretation of the Alexandrov topology of subsets of a simplicial complexes. In Section 7 we present two numerical examples.

2. Preliminaries

In this section we recall some definitions and results needed in the paper and establish some notations.

2.1. **Finite topological spaces.**

We recall that a *topology* on a set X is a family $\mathcal{T}$ of subsets of X which is closed under finite intersection and arbitrary union and satisfies $\varnothing, X \in \mathcal{T}$. The sets in $\mathcal{T}$ are called *open*. The *interior* of A, denoted $\operatorname{int} A$, is the union of open subsets of A. A set $A \subseteq X$ is *closed* if $X \setminus A$ is open. The *closure* of A, denoted $\operatorname{cl} A$, is the intersection of all closed supersets of A. Topology $\mathcal{T}$ is T_2 or Hausdorff if for any $x, y \in X$ where $x \neq y$, there exist disjoint sets $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$. It is T_0 or Kolmogorov if for any $x, y \in X$ such that $x \neq y$ there exists a $U \in \mathcal{T}$ containing precisely one of x and y.

A *topological space* is a pair $(X, \mathcal{T})$ where $\mathcal{T}$ is a topology on X. It is a *finite topological space* if X is finite. Finite topological spaces differ from general topological spaces because the only Hausdorff topology on a finite topological space X is the discrete topology consisting of all subsets of X.

Given two topological spaces $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ we say that a map $f : (X, \mathcal{T}) \to (X', \mathcal{T}')$ is *continuous* if $U \in \mathcal{T}'$ implies $f^{-1}(U) \in \mathcal{T}$.

A remarkable feature of finite topological spaces is the following theorem.

Theorem 2.1. *(P. Alexandrov, [2]) For a preorder $\leq$ on a finite set X, there is a topology $\mathcal{T}_{\leq}$ on X whose open sets are upper sets with respect to $\leq$ that is sets $T \subseteq X$ such that $x \in T$, $x \leq y$ implies $y \in T$. For a topology $\mathcal{T}$ on a finite set X, there is a preorder $\leq_{\mathcal{T}}$ where $x \leq_{\mathcal{T}} y$ if and only if x is in the closure of y with respect to $\mathcal{T}$. The correspondences $\mathcal{T} \mapsto \leq_{\mathcal{T}}$ and $\leq \mapsto \mathcal{T}_{\leq}$ are mutually inverse. They transform continuous maps into an order-preserving maps and vice versa. Moreover, topology $\mathcal{T}$ is T_0 (Kolmogorov) if and only if the preorder $\leq_{\mathcal{T}}$ is a partial order.*

The space X is $\mathcal{T}$-*disconnected* if there exist disjoint, non-empty sets $U, V \in \mathcal{T}$ such that $X = U \cup V$. The space X is $\mathcal{T}$-*connected* if it is not $\mathcal{T}$-disconnected. A subset $A \subseteq X$ is $\mathcal{T}$-*connected* if it is connected as a space with induced topology $\mathcal{T}_A$. The connected component of $x \in X$, denoted $[x]_{\mathcal{T}}$, is the union of all connected subsets of X containing x. Note, that $[x]_{\mathcal{T}}$ is a connected set and $\{\, [x]_{\mathcal{T}} \mid x \in X \,\}$ is a partition of X.

A preorder X is *order-connected* if for any two points $x, y \in X$ there exists a sequence $x_0, x_1, \ldots, x_n$ of points in X, starting in x and ending in y, such that any two consecutive points are comparable.

Proposition 2.2. *(*[4, Proposition 1.2.4]*) Let $(X, \mathcal{T})$ be a finite topological space. Then, the following conditions are equivalent:*

(i) *X is a connected topological space.*
(ii) *X is order-connected with the preorder $\leq_{\mathcal{T}}$.*
(iii) *X is a path-connected topological space.*

□

2.2. **Simplicial complexes as finite topological spaces.** Let K be a finite simplicial complex, either a geometric simplicial complex in $\mathbb{R}^d$ (see [29, Section 1.2]) or an abstract simplicial complex (see [29, Section 1.3]). We consider K as a poset $(K, \preceq)$ with $\sigma \preceq \tau$ if and only if σ is a face of τ (also phrased τ is a *coface* of σ). We define an *interval* between σ and σ', denoted by $[\sigma, \sigma']$, as a set $\{\tau \in K \mid \sigma \preceq \tau \preceq \sigma'\}$. The poset structure of K provides, via Theorem 2.1 (Alexandrov Theorem), a T_0 topology on K. We denote it $\mathcal{T}_K$ and we refer to $\mathcal{T}_K$ as the *Alexandrov topology* of K. We note that $\mathcal{T}_K$ is non-Hausdorff unless K consists of vertices only. However, $\mathcal{T}_K$ is always T_0.

It is easy to see that a set $A \subseteq K$ is closed in the Alexandrov topology if and only if all faces of any element of A are also in A. Hence, the closure of A is the collection of all faces of elements in A.

The non-Hausdorff topology $\mathcal{T}_K$ of a simplicial complex K should not be confused with the topology of the *polytope* $|K|$ of K. In the case of a geometric simplicial complex, the polytope $|K|$ is just the union of all simplices in K. In the case of an abstract simplicial complex, the polytope $|K|$ is defined up to a homeomorphism as the polytope of a geometric realization of K (see [29, Sec. 1.2,1.3]). Polytope $|K|$ is a subset of the Euclidean space with metric topology, therefore its topology is Hausdorff.

An *open cell* $\overset{\circ}{\sigma}$ associated with a simplex $\sigma \in K$ is the set of points x in the polytope $|K|$ whose barycentric coordinates $t_v(x)$ are strictly positive for every vertex $v \in \sigma$. The *solid* of a set of simplices $A \subseteq K$ is $|A| := \bigcup \{\, \overset{\circ}{\sigma} \mid \sigma \in A \,\}$. Note that A is a subcomplex of K if and only if A is closed in the Alexandrov topology of K and then the solid of A coincides with the polytope of A. This is why we use $|\cdot|$ to denote both solids and polytopes. It is not difficult to verify that $A \subseteq K$ is open (respectively closed) in the Alexandrov topology if and only if its solid is open (respectively closed) in $|K|$.

In the case of a geometric simplicial complex we say that a set of simplices $A \subseteq K$ is *convex* if its solid $|A|$ is a convex set in $\mathbb{R}^d$. If the geometric simplicial complex K is convex and $A \subseteq K$ then we define the convex hull of A as the intersection of all convex supersets of A in K. We denote the convex hull of A by $\operatorname{co} A$.

2.3. **Multivalued maps and multivalued dynamics.** Recall that a *multivalued map* $F : X \multimap Y$ is a map which assigns to every point $x \in X$ a non-empty set $F(x) \subseteq Y$. Given

$A \subseteq X$, the *image* of A under F is defined by

$$F(A) := \bigcup_{x \in A} F(x).$$

For the sake of this paper we define a *multivalued dynamical system* in a topological space X as a multivalued map $F : X \times \mathbb{N} \multimap X$ such that

$$F(F(x,m),n) = F(x,n+m). \tag{1}$$

Typically, one also assumes that F is continuous in some sense but we do not need such an assumption in this paper.

Let F be a multivalued dynamical system. Consider the multivalued map $F^n : X \multimap X$ given by $F^n(x) := F(x,n)$. We call F^1 the *generator* of the dynamical system F. It follows from (1) that the multivalued dynamical system F is uniquely determined by the generator. Thus, it is natural to identify a multivalued dynamical system with its generator. In particular, we consider any multivalued map $F : X \multimap X$ as a multivalued dynamical system $F : X \times \mathbb{N} \multimap X$ defined recursively by

$$\begin{aligned} F(x,1) &:= F(x) \\ F(x,n+1) &:= F(F(x,n)). \end{aligned}$$

3. Combinatorial dynamics

In this section we introduce the concept of a combinatorial dynamical system and define solutions and invariant sets of a combinatorial dynamical system. We also present two cases for constructing combinatorial dynamical systems from data.

3.1. **Combinatorial dynamical systems.**

The central object of interest of this paper is given by the following definition.

Definition 3.1. By a *combinatorial dynamical system* we mean a multivalued dynamical system generated by a multivalued map $F : X \multimap X$ from a finite topological space X into itself.

In the sequel we identify the combinatorial dynamical system with its generator. Although in this paper we restrict the considered examples to the case of combinatorial dynamical systems generated by multivalued maps on the collection of simplices of a simplicial complex with its Alexandrov topology, the theoretical results apply to the general setting of finite topological spaces. The general setting of a finite topological space is useful, because there are methods to represent combinatorially subsets of $\mathbb{R}^d$ other than the polytope of a simplicial complex, for instance a cubical complex or a more general cellular complex. All these cases lead to a finite topological space. As we already mentioned in Section 2.3, we do not require any continuity conditions on F. Surprisingly, although such conditions are needed to define the Conley index (see [28]), they are not needed to define the isolating neighborhood and Morse decomposition.

3.2. **Solutions and invariant sets.** A *solution* of F in $A \subseteq K$ is a partial map $\rho : \mathbb{Z} \nrightarrow A$ whose *domain*, denoted $\operatorname{dom}\rho$, is either the set of all integers or a finite interval of integers and for any $i, i+1 \in \operatorname{dom}\rho$ the inclusion $\rho(i+1) \in F(\rho(i))$ holds. The solution ρ is *full* if $\operatorname{dom}\rho = \mathbb{Z}$, otherwise it is *partial*. In the latter case, if $\operatorname{dom}\rho = \mathbb{Z} \cap [m, n]$ for some $m, n \in \mathbb{Z}$, then $\rho(m)$ and $\rho(n)$ are called the left and right *endpoint* of ρ, respectively. The solution *passes* through $\sigma \in K$ if $\sigma = \rho(i)$ for some $i \in \operatorname{dom}\rho$. The set A is *invariant* if for every $\sigma \in A$ there exists a full solution in A passing through σ.

3.3. **A combinatorial dynamical system from a sampled map.** Assume K is a convex simplicial complex in $\mathbb{R}^d$ and consider a map $f : |K| \to |K|$ on the polytope of K. Moreover, assume we know only a noisy sample of f, that is a non-empty collection of pairs $\{\, (x_i, y_i) \,\}_{i=1}^n$ satisfying $x_i, y_i \in |K|$ and y_i equals $f(x_i)$ perturbed by some noise. Our goal is to investigate the dynamical system generated by f on $|K|$ by studying a multivalued dynamical system induced on the finite collection of simplices of K by a multivalued map $F : K \multimap K$ constructed from the sample. In order to construct F recall that a *maximal simplex* or *toplex* in K is a simplex which is not a proper face of another simplex in K. Denote by K_{top} the family of all toplexes in K and assume that each toplex is d-dimensional. For toplexes τ, τ' let $n_{\tau,\tau'}$ denote the number of pairs (x_i, y_i) such that $x_i \in \operatorname{cl}|\tau|$ and $y_i \in \operatorname{cl}|\tau'|$. Set $n_{max} := \max\{\, n_{\tau,\tau'} \mid \tau, \tau' \in K_{top} \,\}$ and let

$$\bar{n}_{\tau,\tau'} := \frac{n_{\tau,\tau'}}{n_{max}}$$

denote the relative frequency. For a threshold μ we first assign to each toplex τ the family

$$A_{\mu,\tau} := \{\, \tau' \in K_{top} \mid \bar{n}_{\tau,\tau'} \geq \mu \,\}$$

that is the collection of toplexes τ' for which the relative frequency $\bar{n}_{\tau,\tau'}$ exceeds the threshold μ. Note that when the map f is strongly expanding and the number of sample points in τ is small, it may happen that some or even all toplexes in $A_{\mu,\tau}$ are disjoint. This is in contrast to the fact that a continuous map sends a connected set to a connected set. We remedy the problem by replacing $A_{\mu,\tau}$ with $\operatorname{co} A_{\mu,\tau}$, the convex hull of $A_{\mu,\tau}$. Next, we extend the definition to a multivalued map $F_\mu : K \multimap K$ by setting, for an arbitrary simplex $\sigma \in K$ (not necessarily a toplex),

$$F_\mu(\sigma) := \operatorname{co} \bigcup \{\, A_{\mu,\tau} \mid \ \sigma \text{ is a face of a toplex } \tau \,\}.$$

The multivalued map $F := F_\mu$ is an example of the generator of a combinatorial dynamical system on the set of simplices of the simplicial complex K.

3.4. **A digraph interpretation of a combinatorial dynamical system.** A combinatorial dynamical system F may be viewed as a digraph G_F whose vertices are simplices in K with a directed edge from σ to τ if and only if $\tau \in F(\sigma)$. An example is presented in Figure 1. The polytope is the interval $[0, 1] \subseteq \mathbb{R}$. The simplicial complex K (see Figure 1(bottom left)) consists of two toplexes **AB** and **BC** and three vertices **A**, **B**, **C** where **A**, **B**, **C** are points in $\mathbb{R}$ with coordinates 0, $\frac{1}{2}$ and 1, respectively. A map $f : [0, 1] \to [0, 1]$ and a noisy sample of this map are presented in Figure 1(left). The relative freqencies are $\bar{n}_{AB,AB} = \frac{11}{12}$,

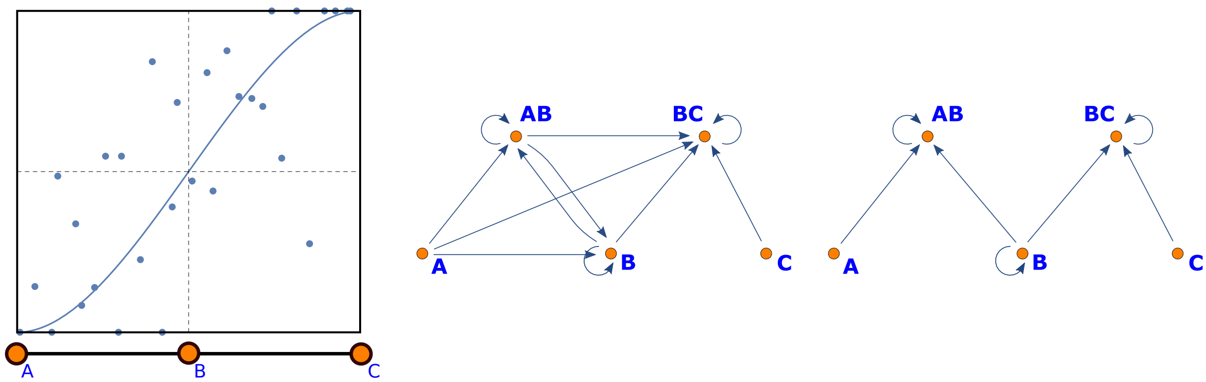

FIGURE 1. Bottom left: A simplicial complex in $\mathbb{R}$ whose polytope is the interval $[0,1]$. Left: A map $f : [0,1] \ni x \mapsto 3x^2 - 2x^3 \in [0,1]$ and a sample of f with large Gaussian noise. Middle and right: The constructed combinatorial dynamical system F_μ with threshold $\mu = 0.3$ (middle) and $\mu = 0.4$ (right).

$\bar{n}_{AB,BC} = \frac{4}{12}$, $\bar{n}_{BC,AB} = \frac{3}{12}$ and $\bar{n}_{BC,BC} = 1$. Figure 1(middle) and Figure 1(right) show digraph presentations of two combinatorial dynamical systems on K respectively for thresholds $\mu = 0.3$ and $\mu = 0.4$. In order to explain the presence of the loop at vertex $\mathbf{B}$ notice that $\mathbf{B}$ is a face of two toplexes: $\mathbf{AB}$ and $\mathbf{BC}$. For thresholds $\mu \in \{0.3, 0.4\}$ we have $\mathbf{AB} \in A_{\mu,\mathbf{AB}}$ and $\mathbf{BC} \in A_{\mu,\mathbf{BC}}$. Therefore, $F_\mu(\mathbf{B}) = \mathrm{co}\{\mathbf{AB}, \mathbf{BC}\} = \{\mathbf{AB}, \mathbf{B}, \mathbf{BC}\}$. But, an analogous computation for vertex $\mathbf{A}$ gives $F_\mu(\mathbf{A}) = \mathrm{co}\{\mathbf{AB}\} = \{\mathbf{AB}\}$, which means that there is no loop at vertex $\mathbf{A}$. Similarly, we see that there is no loop at vertex $\mathbf{C}$.

The digraph interpretation of a combinatorial dynamical system means that some concepts in dynamics may be translated into concepts in digraphs and vice versa. In this translation a solution to F in $A \subseteq K$ corresponds to a walk in G_F through vertices in A and the set A is invariant if every vertex in A is incident to a bi-infinite walk in G_F through vertices in A. For instance, in Figure 1(middle), the set $\{\mathbf{AB}, \mathbf{B}, \mathbf{BC}\}$ is invariant. Actually, all its subsets are also invariant because of the presence of loops at $\mathbf{AB}$, $\mathbf{B}$ and $\mathbf{BC}$. The same comment applies to Figure 1(right).

We emphasize that, despite the convenience of the language of digraphs, the combinatorial dynamical system F is more than just the digraph G_F, because the collection of simplices K, that is the set of vertices of G_F, is a T_0 topological space. In particular, the concept of isolating neighborhood which we define in Section 4.1, cannot be formulated in the language of digraphs only.

3.5. **A combinatorial dynamical system from a sampled vector field.** When the dynamics which is sampled constitutes a flow, that is when time is continuous as in the case of a differential equation, the sampled data often consists of a cloud of points with a vector attached to every point. In this case the construction of combinatorial dynamical system is done in two steps. In the first step the cloud of vectors is transformed into a combinatorial vector field in the sense of Forman [17, 18] or its generalized version of combinatorial multivector field [27]. We discuss one of the possible algorithms for the first step in Section 7.2. In the second step, the combinatorial multivector field is transformed into a combinatorial dynamical system. In order to explain the second step, we introduce some definitions. Let K be a simplicial complex. We say that $A \subseteq K$ is *orderly convex* if for any $\sigma_1, \sigma_2 \in A$ and $\tau \in K$ the relations $\sigma_1 \preceq \tau$ and $\tau \preceq \sigma_2$ imply $\tau \in A$. We remark that orderly convex sets in K may be characterized in the language of the associated Alexandrov topology. Namely, $A \subseteq K$ is orderly convex if and only if it is locally closed (see [16, Sec. 2.7.1, pg 112]) in the Alexandrov topology $\mathcal{T}_K$. We define a *multivector* as an orderly convex subset of K and a *combinatorial multivector field* on K (*combinatorial multivector field* in short) as a partition $\mathcal{V}$ of K into multivectors. Note that this definition of a combinatorial multivector field is less restrictive than the one in [27]. Both definitions encompass the combinatorial vector field of Forman as a special case. The definition of combinatorial multivector field in [27] additionally requires that multivectors have a unique maximal element. This is not needed here.

Given a combinatorial multivector field $\mathcal{V}$, we denote by $[\sigma]_{\mathcal{V}}$ the unique V in $\mathcal{V}$ such that $\sigma \in V$. We associate with $\mathcal{V}$ a combinatorial dynamical system $F_{\mathcal{V}} : K \multimap K$ given by $F_{\mathcal{V}}(\sigma) := \operatorname{cl} \sigma \cup [\sigma]_{\mathcal{V}}$. Note that $F_{\mathcal{V}}$ in general admits more solutions than $\Pi_{\mathcal{V}}$ defined in [27, Section 5.4]. In particular, each $\sigma \in K$ is a *fixed point* of $F_{\mathcal{V}}$, that is, $\sigma \in F_{\mathcal{V}}(\sigma)$. This may look like a drawback but actually it simplifies the theory and allows detecting and eliminating spurious fixed points by the triviality of their Conley index [21].

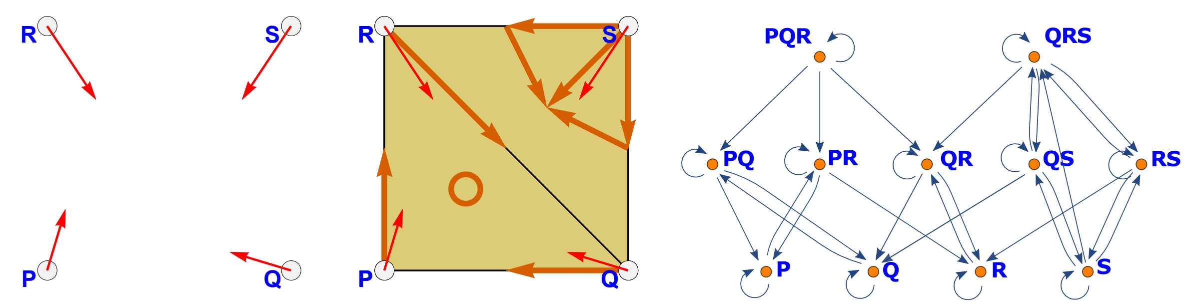

FIGURE 2. Left: A cloud of vectors. Middle: A possible combinatorial multivector field representation of the cloud of vectors. Right: The associated combinatorial dynamical system represented as a digraph.

Figure 2(left) presents a toy example of a cloud of vectors. It consists of four vectors marked red at four points $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{S}$. One of possible geometric simplicial complexes with vertices at points $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{S}$ is the simplicial complex K consisting of triangles $\mathbf{PQR}$, $\mathbf{QRS}$ and its faces. A possible multivector field $\mathcal{V}$ on K constructed from the cloud of vectors consists of multivectors $\{\mathbf{P}, \mathbf{PR}\}$, $\{\mathbf{R}, \mathbf{QR}\}$, $\{\mathbf{Q}, \mathbf{PQ}\}$, $\{\mathbf{PQR}\}$,$\{\mathbf{S}, \mathbf{RS}, \mathbf{QS}, \mathbf{QRS}\}$. It is indicated in Figure 2(middle) by orange arrows between centers of mass of simplices. Note that in order to keep the figure legible, only arrows in the direction increasing the dimension are marked. The singleton $\{\mathbf{PQR}\}$ is marked with an orange circle. The associated combinatorial dynamical system $F_{\mathcal{V}}$ presented as a digraph is in Figure 2(right). Note that in general K and $\mathcal{V}$ are not uniquely determined by the cloud of vectors. One possible method for constructing combinatorial multivector fields from a cloud of vectors is discussed in Section 7.2.

4. Isolated invariant sets and Morse decompositions

In this section we consider a combinatorial dynamical system $F : X \multimap X$ and define for F the concepts of isolating neighborhood, isolated invariant set and Morse decomposition.

4.1. **Isolated invariant sets.** The closed set $N \subseteq K$ is an *isolating neighborhood* for an invariant set $S \subseteq K$ if S is contained in N and any partial solution in N with endpoints in S has all values in S. If such an isolating neighborhood for S exists, we say that S is an *isolated invariant set*. We emphasize that, unlike the classical theory, the same set N may be an isolating neighborhood for more than one isolated invariant set.

The invariant set $\{\mathbf{AB}\}$ in Figure 1(middle) is not an isolated invariant set, because for any closed set N containing $\{\mathbf{AB}\}$ the partial solution $(\mathbf{AB}, \mathbf{B}, \mathbf{AB})$ is contained in N and has endpoints in $\{\mathbf{AB}\}$. The invariant sets $\{\mathbf{BC}\}$ and $\{\mathbf{AB}, \mathbf{B}\}$ are both isolated invariant sets and $\{\mathbf{A}, \mathbf{AB}, \mathbf{B}, \mathbf{BC}, \mathbf{C}\}$ is an isolating neighborhood for both.

Since we have a loop at every vertex of the digraph of the combinatorial dynamical system in Figure 2(right), every set is invariant. In particular, every singleton is invariant. However, the only singleton which is an isolated invariant set is $\{\mathbf{PQR}\}$. Its isolating neighborhood is $\operatorname{cl}\{\mathbf{PQR}\} = \{\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{PQ}, \mathbf{PR}, \mathbf{QR}, \mathbf{PQR}\}$. Another isolated invariant set with the same isolating neighborhood is $\{\mathbf{PQ}, \mathbf{PR}, \mathbf{QR}\}$.

The *maximal invariant set* of F, denoted $S(F)$, is the set of all simplices $\sigma \in K$ such that there exists a full solution of F in K passing through σ. It is straightforward to observe that $S(F)$ is invariant and K is an isolating neighborhood for $S(F)$. Therefore, $S(F)$ is an isolated invariant set. Note that the maximal invariant set $S(F_{\mathcal{V}})$ for a combinatorial multivector field $\mathcal{V}$ is always the whole K, because for each $\sigma \in K$ we have $\sigma \in \operatorname{cl}\sigma \subseteq F_{\mathcal{V}}(\sigma)$. This is visible in Figure 2(right) as a loop at every vertex. In contrast, $\mathbf{A}$ does not belong to the maximal invariant set in Figure 1(right).

4.2. **Morse decompositions.** A *directed connection*, or briefly a *connection*, from an isolated invariant set S_1 to an isolated invariant set S_2 is a partial solution with left endpoint in S_1 and right endpoint in S_2. A family $\mathcal{M}$ consisting of mutually disjoint, non-empty isolated invariant subsets of an isolated invariant set S is a *Morse decomposition* of S if $\mathcal{M}$ admits a partial order $\leq$ such that any connection between elements in $\mathcal{M}$ either has all values in a single element of $\mathcal{M}$ or it originates in $M \in \mathcal{M}$ and terminates in M' such that $M > M'$. If S is not mentioned explicitly, we mean a Morse decomposition of the maximal invariant set $S(F)$. The elements of $\mathcal{M}$ are called *Morse sets*. Although the definitions of isolated invariant set and Morse decomposition require topology, there is an important case when they correspond to purely graph-theoretic concepts. An isolated invariant set is *minimal* if it admits no non-trivial Morse decomposition that is no Morse decomposition consisting of more than one Morse set. A Morse decomposition is *minimal* if each of its Morse sets is minimal. The following theorem shows that the minimal Morse decomposition of F, denoted as $\mathcal{M}(F)$, is unique and consists of the strongly connected components of G_F.

Theorem 4.1. *The family of all strongly connected components of G_F is the unique minimal Morse decomposition of $S(F)$.*

Proof: Let $\mathcal{S}$ be the family of all strongly connected components of G_F. We will show that K is an isolating neighborhood for any $S \in \mathcal{S}$. Obviously, S, as a strongly connected component, is invariant. And K, as the whole space, is closed. Therefore, $S \subseteq K$ is an isolated invariant set. Moreover, any partial solution with endpoints in S must have all values in S, because S is a strongly connected component of G_F. Hence, each $S \in \mathcal{S}$ is an isolated invariant set. Clearly, it is a minimal isolated invariant set. For $S_1, S_2 \in \mathcal{S}$ we write $S_1 \geq S_2$ if there exists a directed connection from S_1 to S_2. Since $\mathcal{S}$ consists of strongly connected components, this defines a partial order on $\mathcal{S}$. Let ρ be a connection from S_1 to S_2 whose values are not contained in a single element of $\mathcal{S}$. Then, $S_1 > S_2$. This proves that $\mathcal{S}$ is a Morse decomposition. Obviously, a strongly connected component cannot have a non-trivial Morse decomposition. Thus, $\mathcal{S}$ is a minimal Morse decomposition. Assume that $\mathcal{S}'$ is another minimal Morse decomposition and $S' \in \mathcal{S}'$. We claim that S' is strongly connected as a subgraph of G_F. Indeed, if not, then, according to what we already proved, the strongly connected components of S' would constitute a non-trivial Morse decomposition of S', contradicting the assumption that S' is a minimal invariant set. Hence, S' is contained in a Morse set $S \in \mathcal{S}$. By a symmetric argument every $S \in \mathcal{S}$ is contained in a Morse set $S' \in \mathcal{S}'$. This shows that $\mathcal{S} = \mathcal{S}'$ and proves the uniqueness. $\square$

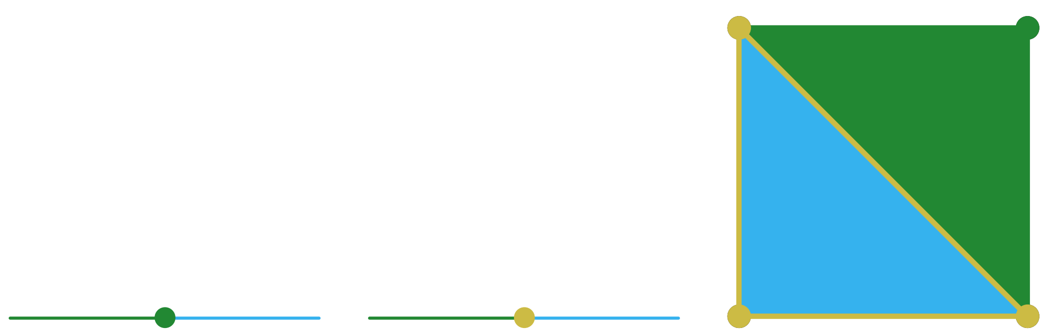

FIGURE 3. Left: The minimal Morse decomposition of the combinatorial dynamical system in Figure 1(middle) consisting of $\{\mathbf{AB}, \mathbf{B}\}$ (green) and $\{\mathbf{BC}\}$ (blue). Middle: The minimal Morse decomposition of the combinatorial dynamical system in Figure 1(right) consisting of $\{\mathbf{AB}\}$ (green), $\{\mathbf{B}\}$ (yellow) and $\{\mathbf{BC}\}$ (blue). Right: The minimal Morse decomposition of the combinatorial dynamical system in Figure 2(right) consisting of $\{\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{PQ}, \mathbf{PR}, \mathbf{QR}\}$ (yellow), $\{\mathbf{S}, \mathbf{RS}, \mathbf{QS}, \mathbf{QRS}\}$ (green) and $\{\mathbf{PQR}\}$ (blue).

Consider the combinatorial dynamical system in Figure 1(middle). Its minimal Morse decomposition consists of two Morse sets: $\{\mathbf{AB}, \mathbf{B}\}$ and $\{\mathbf{BC}\}$ with $\{\mathbf{AB}, \mathbf{B}\} > \{\mathbf{BC}\}$. The minimal Morse decomposition of the combinatorial dynamical system in Figure 1(right) consists of three Morse sets: $\{\mathbf{AB}\}$, $\{\mathbf{B}\}$ and $\{\mathbf{BC}\}$ with $\{\mathbf{B}\} > \{\mathbf{BC}\}$ and $\{\mathbf{B}\} > \{\mathbf{AB}\}$. The minimal Morse decomposition of the combinatorial dynamical system in Figure 2(right) consists of three isolated invariant sets: $M_1 := \{\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{PQ}, \mathbf{PR}, \mathbf{QR}\}$, $M_2 := \{\mathbf{S}, \mathbf{RS}, \mathbf{QS}, \mathbf{QRS}\}$ and $M_3 := \{\mathbf{PQR}\}$ with $M_3 > M_1$ and $M_2 > M_1$. These minimal Morse decompositions are illustrated in Figure 3.

5. Persistence of Morse decompositions.

In this section we define homological persistence of Morse decompositions in the setting of combinatorial dynamical systems.

5.1. **Disconnecting topology.** In the classical setting of semiflows on locally compact Hausdorff (T_2) topological spaces, isolated invariant sets are always compact. Therefore, given a Morse decomposition in such a classical setting which consists of more than one Morse set, the union of all Morse sets is always disconnected in the topology induced from the space. This is because Morse sets are always disjoint and in this case also closed as compact sets in a Hausdorff space. In particular, the space between the Morse sets is filled with solutions connecting the Morse sets. But, in finite topological spaces the Morse sets need not be closed and solutions may jump directly from one Morse set to another Morse set. Consequently, the union of Morse sets generally is not disconnected. Thus, we need a method to disconnect Morse sets. Fortunately, in this case we do not need space between the Morse sets. We achieve the separation by purely topological methods. To explain this, we need the following terminology, notation and theorem.

Assume $\mathcal{A}$ is a finite family of mutually disjoint non-empty sets and $\mathcal{T}$ is a topology on $\bigcup \mathcal{A}$. We say that $\mathcal{A}$ is *disconnected* in $\mathcal{T}$ if each set $A \in \mathcal{A}$ is open in the topology $\mathcal{T}$.

Given a family $\mathcal{A}$ of subsets of a set X, we use the notation $\mathcal{A}^* := \{\bigcup \mathcal{A}' : \mathcal{A}' \subseteq \mathcal{A}\}$ for the smallest family of sets in X, containing $\mathcal{A}$ and closed under summation. If $\mathcal{B}$ is another such family, we write $\mathcal{A} \bar{\cap} \mathcal{B}$ for the family of intersections of every set in $\mathcal{A}$ with every set in $\mathcal{B}$. We say that $\mathcal{A}$ is *inscribed* in $\mathcal{B}$ and write $\mathcal{A} \sqsubset \mathcal{B}$, if for every $A \in \mathcal{A}$ there exists a $B \in \mathcal{B}$, such that $A \subseteq B$.

In order to shorten the notation we will also write $\langle \mathcal{A} \rangle$ for the union $\bigcup \mathcal{A}$ of all the sets in $\mathcal{A}$. Note that if $A \subseteq X$ and $\mathcal{T}$ is a topology on X, then the topology induced by $\mathcal{T}$ on A is $A \bar{\cap} \mathcal{T} := \{A\} \bar{\cap} \mathcal{T}$.

Theorem 5.1. *Assume $(X, \mathcal{T})$ is an arbitrary topological space and $\mathcal{A}$ is a finite family of mutually disjoint, non-empty subsets of X. Then $\mathcal{T}_{\mathcal{A}} := (\mathcal{A} \bar{\cap} \mathcal{T})^*$ is a topology on $\langle \mathcal{A} \rangle$. Moreover,*

- (i) *if $\mathcal{T}$ is a T_0 topology, then so is $\mathcal{T}_{\mathcal{A}}$,*
- (ii) *for every $A \in \mathcal{A}$, the topology induced on A by $\mathcal{T}$ coincides with the topology induced on A by $\mathcal{T}_{\mathcal{A}}$,*
- (iii) *the family $\mathcal{A}$ is $\mathcal{T}_{\mathcal{A}}$-disconnected,*

(iv) *if additionally $\langle\mathcal{A}\rangle = X$ and each set in $\mathcal{A}$ is $\mathcal{T}$-connected, then the connected components with respect to $\mathcal{T}_{\mathcal{A}}$ coincide with the sets in $\mathcal{A}$.*

Proof: We will show that $\mathcal{A} \bar{\cap} \mathcal{T}$ is a basis (see [30, Section 13]) for some topology on $\langle\mathcal{A}\rangle$. Let $x \in \langle\mathcal{A}\rangle$. There exists an $A \in \mathcal{A}$ such that $x \in A$. Hence, $x \in A = A \cap X \in \mathcal{A} \bar{\cap} \mathcal{T}$. Assume that $x \in (A \cap U) \cap (B \cap V)$ for some $A, B \in \mathcal{A}$ and $U, V \in \mathcal{T}$. Then $A = B$ and consequently $(A \cap U) \cap (B \cap V) = A \cap (U \cap V) \in \mathcal{A} \bar{\cap} \mathcal{T}$. This shows that $\mathcal{A} \bar{\cap} \mathcal{T}$ is indeed a basis. By [30, 13.1] it follows that $\mathcal{T}_{\mathcal{A}}$ is a topology. For the proof of (i) consider $x, y \in \langle\mathcal{A}\rangle$, $x \neq y$. Without loss of generality we can assume that there exists an open neighbourhood $U \in \mathcal{T}$ of y such that $x \notin U$. Let $A \in \mathcal{A}$ be such that $y \in A$. Then $y \in U \cap A$ and $x \notin U \cap A$, hence (i) holds. To prove (ii) we need to show that $A \bar{\cap} \mathcal{T} = A \bar{\cap} \mathcal{T}_{\mathcal{A}}$. Obviously $A \bar{\cap} \mathcal{T} \subseteq A \bar{\cap} \mathcal{T}_{\mathcal{A}}$. To prove the opposite inclusion take a $V \in A \bar{\cap} \mathcal{T}_{\mathcal{A}}$. This means that there is a $U \in \mathcal{T}_{\mathcal{A}}$ such that $V = A \cap U$. Then $U = \bigcup_{i \in I}(A_i \cap U_i)$ for some $U_i \in \mathcal{T}$ and $A_i \in \mathcal{A}$. Hence, $V = A \cap U = A \cap \bigcup_{i \in I}(A_i \cap U_i) = A \cap \bigcup_{i \in I_A} U_i \in A \bar{\cap} \mathcal{T}$ where $I_A = \{i \in I : A_i = A\}$ and (ii) is proved. Property (iii) is obvious, because $A \in \mathcal{A}$ implies $A = A \cap X \in \mathcal{T}_{\mathcal{A}}$. To prove (iv) assume A is $\mathcal{T}$-connected. It follows from (ii) that A is $\mathcal{T}_{\mathcal{A}}$-connected. Let $x \in A$. Then A is contained in $[x]_{\mathcal{T}_{\mathcal{A}}}$, the $\mathcal{T}_{\mathcal{A}}$-connected component of x. This means that $[x]_{\mathcal{T}_{\mathcal{A}}} = \bigcup \mathcal{A}'$ for some $\mathcal{A}' \subseteq \mathcal{A}$. Since every set in $\mathcal{A}$ is open in $\mathcal{T}_{\mathcal{A}}$, the family $\mathcal{A}'$ must contain precisely one element. Consequently $A = [x]_{\mathcal{T}_{\mathcal{A}}}$ and (iv) holds. □

We note that given a Morse decomposition $\mathcal{M}$ of a combinatorial dynamical system F on a finite simplicial complex K, in general the union $\langle\mathcal{M}\rangle$ is not a subcomplex of the simplicial complex K. Therefore, we cannot take simplicial homology of $\langle\mathcal{M}\rangle$. Moreover, we are interested in the special topology $\mathcal{T}_{\mathcal{M}}$ on $\langle\mathcal{M}\rangle$ where $\mathcal{T}$ is the Alexandrov topology of K. The topology $\mathcal{T}_{\mathcal{M}}$ separates the Morse sets due to Theorem 5.1(iii). Fortunately, the singular homology makes sense for any topological space, in particular we can consider $H(\langle\mathcal{M}\rangle, \mathcal{T}_{\mathcal{M}})$. In Section 6 we use McCord's Theorem [23] to show that $H(\langle\mathcal{M}\rangle, \mathcal{T}_{\mathcal{M}})$ may be computed as simplicial homology of a subcomplex of the barycentric subdivision of K.

5.2. **Persistence and zigzag persistence of Morse decompositions.** Consider two simplicial complexes K and K' with combinatorial dynamical systems F on K and F' on K' and a map $f : K \to K'$ continuous with respect to Alexandrov topologies $\mathcal{T}$ on K and $\mathcal{T}'$ on K'. By Theorem 2.1 (Alexandrov Theorem) the map f is continuous if and only if it preserves the face relation in K and K'. In particular, every simplicial map is continuous.

The following theorem lets us define homomorphisms in homology needed to set up persistence of Morse decompositions.

Theorem 5.2. *Let $\mathcal{M}$ and $\mathcal{M}'$ be Morse decompositions respectively for F and F'. Assume that a continuous map $f : K \to K'$* respects *$\mathcal{M}$ and $\mathcal{M}'$, that is $f(\mathcal{M}) \sqsubset \mathcal{M}'$ where $f(\mathcal{M}) := \{\, f(M) \mid M \in \mathcal{M} \,\}$. Then, the map $f_{\mathcal{M},\mathcal{M}'} : (\langle\mathcal{M}\rangle, \mathcal{T}_{\mathcal{M}}) \ni \sigma \mapsto f(\sigma) \in (\langle\mathcal{M}'\rangle, \mathcal{T}'_{\mathcal{M}'})$ is well defined and continuous.*

Proof: Let $\sigma \in \langle\mathcal{M}\rangle$. Then $\sigma \in M$ for some $M \in \mathcal{M}$. Since f respects $\mathcal{M}$ and $\mathcal{M}'$, there is an $M' \in \mathcal{M}'$ such that $f(M) \subseteq M'$. It follows that $f(\sigma) \in \langle\mathcal{M}'\rangle$. Hence, $f_{\mathcal{M},\mathcal{M}'}$ is well defined. Since $\mathcal{M}' \bar{\cap} \mathcal{T}'$ is a basis of topology $\mathcal{T}'_{\mathcal{M}'}$, in order to prove continuity it suffices to show that for any $M' \in \mathcal{M}'$ and $T' \in \mathcal{T}'$ the set $f^{-1}_{\mathcal{M},\mathcal{M}'}(M' \cap T')$ is open in $\mathcal{T}_{\mathcal{M}}$. Let $\mathcal{M}_{M'} := \{ M \in \mathcal{M} \mid f(M) \subseteq M' \}$. Then $f^{-1}(M') \cap \langle \mathcal{M} \rangle = \langle \mathcal{M}_{M'} \rangle$. By continuity of f we have $f^{-1}(T') \in \mathcal{T}$. Therefore, $f^{-1}_{\mathcal{M},\mathcal{M}'}(M' \cap T') = f^{-1}(M') \cap f^{-1}(T') \cap \langle \mathcal{M} \rangle = f^{-1}(T') \cap \langle \mathcal{M}_{M'} \rangle = \bigcup \{ M \cap f^{-1}(T') \mid M \in \mathcal{M}_{M'} \} \in (\mathcal{M} \bar{\cap} \mathcal{T})^* = \mathcal{T}_{\mathcal{M}}$, which completes the proof. □

For a minimal Morse decomposition, denoted by $\mathcal{M}(F)$, we have the following corollary.

Corollary 5.3. *The map $f_{\mathcal{M}(F),\mathcal{M}(F')} : (\langle \mathcal{M}(F) \rangle, \mathcal{T}_{\mathcal{M}(F)}) \to (\langle \mathcal{M}(F') \rangle, \mathcal{T}'_{\mathcal{M}(F')})$ is continuous under the assumption that $f \circ F \subseteq F' \circ f$, that is $f(F(\sigma)) \subseteq F'(f(\sigma))$ for any $\sigma \in K$.*

Proof: By Theorem 5.2, it suffices to show that f respects $\mathcal{M}(F)$ and $\mathcal{M}(F')$. Let $M \in \mathcal{M}(F)$. By Theorem 4.1, the Morse set M is a strongly connected component of G_F. Let $\sigma, \tau \in M$ and let ρ be a partial solution in M with endpoints σ and τ. It follows from the assumption that $f \circ \rho$ is a solution in $f(M)$ with endpoints $f(\sigma)$ and $f(\tau)$. Since $\sigma, \tau \in M$ are arbitrary, the set $f(M)$ must be contained in one strongly connected component of $G_{F'}$, that is $f(M) \subseteq M'$ for some $M' \in \mathcal{M}'$. □

Assume now that, for $i = 1, 2, \ldots n$, we have a simplicial complex K_i with Alexandrov topology $\mathcal{T}^i$, a combinatorial dynamical system F_i on K_i, and a Morse decomposition $\mathcal{M}_i$ of F_i. Let $\{f_i : K_i \to K_{i+1}\}_{i=1,n-1}$ be a sequence of continuous maps such that $f_i \circ F_i \subseteq F_{i+1} \circ f_i$ and $f_i(\mathcal{M}_i) \sqsubset \mathcal{M}_{i+1}$. Note that, by Corollary 5.3, the latter condition holds if $\mathcal{M}_i = \mathcal{M}(F_i)$. It follows from Theorem 5.2 that the maps $\bar{f}_i := (f_i)_{\mathcal{M}_i,\mathcal{M}_{i+1}} : (\langle \mathcal{M}_i \rangle, \mathcal{T}^i_{\mathcal{M}_i}) \to (\langle \mathcal{M}_{i+1} \rangle, \mathcal{T}^{i+1}_{\mathcal{M}_{i+1}})$ are continuous. Thus, we have homomorphisms induced in singular homology $H(\bar{f}_i) : H(\langle \mathcal{M}_i \rangle, \mathcal{T}^i_{\mathcal{M}_i}) \to H(\langle \mathcal{M}_{i+1} \rangle, \mathcal{T}^{i+1}_{\mathcal{M}_{i+1}})$. This yields a persistence module

$$H(\langle \mathcal{M}_1 \rangle, \mathcal{T}^1_{\mathcal{M}_1}) \xrightarrow{H(\bar{f}_1)} H(\langle \mathcal{M}_2 \rangle, \mathcal{T}^2_{\mathcal{M}_2}) \xrightarrow{H(\bar{f}_2)} \ldots \xrightarrow{H(\bar{f}_{n-1})} H(\langle \mathcal{M}_n \rangle, \mathcal{T}^n_{\mathcal{M}_n}). \tag{2}$$

We refer to the persistence diagram of this module as the *persistence diagram of Morse decompositions*. We note that zigzag persistence diagram of Morse decompositions may be obtained analogously by replacing, whenever appropriate, inclusions $f_i \circ F_i \subseteq F_{i+1} \circ f_i$ by $f_i \circ F_i \supseteq F_{i+1} \circ f_i$ and respectively $\bar{f}_i(\mathcal{M}_i) \sqsubset \mathcal{M}_{i+1}$ by $\mathcal{M}_i \sqsupset \bar{f}_i(\mathcal{M}_{i+1})$.

5.3. **Persistence in combinatorial multivector fields.** Let $\mathcal{V}$ be a combinatorial multivector field on a simplicial complex K. We say that $\mathcal{M}$ is a Morse decomposition of $\mathcal{V}$ if it is a Morse decomposition of the associated combinatorial dynamical system $F_{\mathcal{V}}$. We extend this terminology to minimal Morse decompositions. We denote the minimal Morse decomposition of $\mathcal{V}$ by $\mathcal{M}(\mathcal{V}) := \mathcal{M}(F_{\mathcal{V}})$ and the topology of this Morse decomposition by $\mathcal{T}_{\mathcal{V}} := \mathcal{T}_{\mathcal{M}(\mathcal{V})}$.

Theorem 5.4. *Morse decompositions of combinatorial multivector fields have the following properties.*

(i) *The minimal Morse decomposition of a combinatorial multivector field $\mathcal{V}$ on K is a partition of K. In particular, $\langle \mathcal{M} \rangle = K$.*

(ii) *Given $\mathcal{W}$, another combinatorial multivector field on K, the family $\mathcal{V}\bar{\cap}\mathcal{W}$ is a combinatorial multivector field on K. It is inscribed both in $\mathcal{V}$ and $\mathcal{W}$. Moreover, If $\mathcal{V}\sqsubset\mathcal{W}$, then $F_{\mathcal{V}}\subseteq F_{\mathcal{W}}$.*

(iii) *If $\mathcal{V}'$ is a combinatorial multivector field on a simplicial complex K' and $f : K \to K'$ is continuous, then $f^*(\mathcal{V}') := \{\, f^{-1}(V') \mid V' \in \mathcal{V}'\,\}$, called the* pullback *of $\mathcal{V}'$, is a combinatorial multivector field on K.*

(iv) *The maps $\kappa := \mathrm{id}_{\mathcal{V}\bar{\cap}f^*(\mathcal{V}'),\mathcal{V}} : (K, \mathcal{T}_{\mathcal{V}\bar{\cap}f^*(\mathcal{V}')}) \to (K, \mathcal{T}_{\mathcal{V}})$ induced by identity and $\lambda := f_{\mathcal{V}\bar{\cap}f^*(\mathcal{V}'),\mathcal{V}'} : (K, \mathcal{T}_{\mathcal{V}\bar{\cap}f^*(\mathcal{V}')}) \to (K', \mathcal{T}'_{\mathcal{V}'})$ induced by f are continuous.*

Proof: Note that by Theorem 4.1, the Morse sets in the minimal Morse decomposition are the strongly connected components of $G_{F_{\mathcal{V}}}$. Hence, to prove (i) it suffices to observe that every $\sigma \in K$ belongs to a strongly connected component. This is obvious because $\sigma \in \mathrm{cl}\,\sigma \subseteq F_{\mathcal{V}}(\sigma)$ for any $\sigma \in K$. Thus, (i) is proved. Since the intersection of two orderly convex sets is easily seen to be orderly convex, each element of $\mathcal{V}\bar{\cap}\mathcal{W}$ is orderly convex. Obviously, $\mathcal{V}\bar{\cap}\mathcal{W}$ is a partition of K and is inscribed in $\mathcal{V}$ and $\mathcal{W}$. Take $\sigma \in K$. Assumption $\mathcal{V}\sqsubset\mathcal{W}$ implies that $[\sigma]_{\mathcal{V}} \subseteq [\sigma]_{\mathcal{W}}$. It follows that $F_{\mathcal{V}}(\sigma) = \mathrm{cl}\,\sigma \cup [\sigma]_{\mathcal{V}} \subseteq \mathrm{cl}\,\sigma \cup [\sigma]_{\mathcal{W}} = F_{\mathcal{W}}(\sigma)$. Thus, (ii) is also proved. Obviously, $f^*(\mathcal{V}')$ is a partition of K. To show that for every $V' \in \mathcal{V}'$ the set $f^{-1}(V')$ is orderly convex, take $\sigma, \sigma' \in f^{-1}(V')$ and $\tau \in K$ such that $\sigma \preceq \tau \preceq \sigma'$. Then $f(\sigma), f(\sigma') \in V'$, $f(\sigma) \preceq f(\tau) \preceq f(\sigma')$, and since V' is orderly convex, we get $f(\tau) \in V'$. It follows that $\tau \in f^{-1}(V')$ and $f^{-1}(V')$ is orderly convex. This proves (iii). To prove (iv), we verify that the maps κ and λ satisfy the assumption of Corollary 5.3. It follows from (ii) that $\mathrm{id} \circ F_{\mathcal{V}\bar{\cap}f^*(\mathcal{V}')} = F_{\mathcal{V}\bar{\cap}f^*(\mathcal{V}')} \subseteq F_{\mathcal{V}} = F_{\mathcal{V}} \circ \mathrm{id}$ which proves that κ is continuous. Similarly, we get $f \circ F_{\mathcal{V}\bar{\cap}f^*(\mathcal{V}')} \subseteq f \circ F_{f^*(\mathcal{V}')}$. Thus, it suffices to prove that $f \circ F_{f^*(\mathcal{V}')} \subseteq F_{\mathcal{V}'} \circ f$. Indeed, for $\sigma \in K$ we get from the continuity of f and the definition of $f^*(\mathcal{V}')$ that $(f \circ F_{f^*(\mathcal{V}')})(\sigma) = f(F_{f^*(\mathcal{V}')}(\sigma)) = f(\mathrm{cl}\,\sigma \cup [\sigma]_{f^*(\mathcal{V}')}) = f(\mathrm{cl}\,\sigma) \cup f([\sigma]_{f^*(\mathcal{V}')}) \subseteq \mathrm{cl}\, f(\sigma) \cup [f(\sigma)]_{\mathcal{V}'} = F_{\mathcal{V}'}(f(\sigma)) = (F_{\mathcal{V}'} \circ f)(\sigma)$. ∎

We use the diagram of continuous maps $(K, \mathcal{T}_{\mathcal{V}}) \xleftarrow{\kappa} (K, \mathcal{T}_{\mathcal{V}\bar{\cap}f^*(\mathcal{V}')}) \xrightarrow{\lambda} (K', \mathcal{T}_{\mathcal{V}'})$, referred to as the *comparison diagram* of combinatorial multivector fields $\mathcal{V}$ and $\mathcal{V}'$, to define the persistence of Morse decompositions for combinatorial multivector fields. To this end, assume that, for $i = 1, 2, \dots n$, we have a combinatorial multivector field $\mathcal{V}_i$ on a simplicial complex K_i. Moreover, assume that we have a sequence of continuous maps $f_i : K_i \to K_{i+1}$. Putting together the comparison diagrams of $\mathcal{V}_i$ and $\mathcal{V}_{i+1}$ and applying the singular homology functor we obtain the following zigzag persistence module (see [10])

$$
\begin{aligned}
(3) \quad & H(K_1, \mathcal{T}^1_{\mathcal{V}_1}) \xleftarrow{H(\kappa_1)} H(K_1, \mathcal{T}^1_{\mathcal{V}_1\bar{\cap}f^*(\mathcal{V}_2)}) \xrightarrow{H(\lambda_1)} H(K_2, \mathcal{T}^2_{\mathcal{V}_2}) \xleftarrow{H(\kappa_2)} \dots \\
& \qquad \dots \xleftarrow{H(\kappa_n)} H(K_{n-1}, \mathcal{T}^{n-1}_{\mathcal{V}_{n-1}\bar{\cap}f^*(\mathcal{V}_n)}) \xrightarrow{H(\lambda_n)} H(K_n, \mathcal{T}^n_{\mathcal{V}_n}).
\end{aligned}
$$

We refer to the persistence diagram of this module (see [15, 10]) as the *persistence diagram of Morse decompositions* of the sequence of combinatorial multivector fields $\mathcal{V}_i$.

6. Computational considerations and a geometric interpretation

In this section we discuss computational aspects of the theory and provide a geometric interpretation of the Alexandrov topology of subsets of a simplicial complexes.

6.1. **Computational considerations.** Singular homology is not very amenable to computations. Therefore, to compute the persistence module (possibly zigzag) in (2) and (3) efficiently, we take a more combinatorial approach. We take the help of Theorem 6.2 (McCord's Theorem) in order to convert (2) and (3) to a persistence module where the objects are simplicial homology groups.

Let $(X, \mathcal{T})$ be a finite T_0 topological space and let $\leq_{\mathcal{T}}$ be the partial order associated with $\mathcal{T}$ by Theorem 2.1 (Alexandrov). The *nerve* of this partial order, that is, the collection of subsets linearly ordered by $\leq_{\mathcal{T}}$ called *chains*, forms an abstract simplicial complex. We denote it $N(X, \mathcal{T})$ or briefly $N(X)$ if $\mathcal{T}$ is clear from the context. Also by Alexandrov Theorem, a continuous map $f : (X, \mathcal{T}) \to (X', \mathcal{T}')$ of two finite topological T_0 spaces preserves the partial orders $\leq_{\mathcal{T}}$ and $\leq'_{\mathcal{T}}$. Therefore, it induces a simplicial map $N(f) : N(X, \mathcal{T}) \to N(X', \mathcal{T}')$. Recall that every continuous and hence simplicial map $f : K \to K'$ of simplicial complexes extends linearly to a continuous map $|f| : |K| \to |K'|$ on the polytopes of K and K' (cf. [29, Lemma 2.7]). The following proposition is straightforward.

Proposition 6.1. *If K is a simplicial complex, then the barycentric subdivision (cf.* [29, Sec. 2.15]*) of a geometric realization of K is a geometric realization of $N(K)$. In particular, $|K| = |N(K)|$. Moreover, if $f : K \to K'$ is continuous, then $|f| = |N(f)|$.*

Consider the map $\mu_{(X,\mathcal{T})} : |N(X, \mathcal{T})| \ni x \mapsto \min \sigma_x \in X$, where σ_x denotes the unique simplex $\sigma \in N(X, \mathcal{T})$ such that $x \in \overset{\circ}{\sigma}$ and the minimum is taken with respect to the partial order $\leq_{\mathcal{T}}$.

Theorem 6.2. *(M. C. McCord,* [23]*) The map $\mu_{(X,\mathcal{T})}$ is continuous and a weak homotopy equivalence. Moreover, if $f : (X, \mathcal{T}) \to (X', \mathcal{T}')$ is a continuous map of two finite T_0 topological spaces, then the following diagrams commute.*

$$
\begin{tikzcd}
{|N(X,\mathcal{T})|} \arrow[r, "|N(f)|"] \arrow[d, "\mu_{(X,\mathcal{T})}"] & {|N(X',\mathcal{T}')|} \arrow[d, "\mu_{(X',\mathcal{T}')}"] \\
(X,\mathcal{T}) \arrow[r, "f"] & (X',\mathcal{T}')
\end{tikzcd}
\qquad
\begin{tikzcd}
H_k(|N(X,\mathcal{T})|) \arrow[r, "H(|N(f)|)"] \arrow[d, "H(\mu_{(X,\mathcal{T})})"] & H_k(N(|X',\mathcal{T}'|)) \arrow[d, "H(\mu_{(X',\mathcal{T}')})"] \\
H_k(X,\mathcal{T}) \arrow[r, "H(f)"] & H_k(X',\mathcal{T}')
\end{tikzcd}
$$

By McCord's Theorem above, there is a continuous map $\mu_{(X,\mathcal{T})} : |N(X, \mathcal{T})| \to X$ which induces an isomorphism $H(\mu_{(X,\mathcal{T})}) : H(|N(X, \mathcal{T})|) \to H(X, \mathcal{T})$ of singular homologies. Moreover, the map $(X, \mathcal{T}) \mapsto H(\mu_{(X,\mathcal{T})})$ is a natural transformation, that is for any continuous map $f : (X, \mathcal{T}) \to (X', \mathcal{T}')$ of finite T_0 topological spaces $H(\mu_{(X',\mathcal{T}')}) \circ H(|N(f)|) = H(f) \circ H(\mu_{(X,\mathcal{T})})$. Applying McCord's Theorem to every homology group in (2) we obtain the following proposition.

Proposition 6.3. *Persistence module* (2) *is isomorphic to the persistence module*

$$H(|N(\langle\mathcal{M}_1\rangle,\mathcal{T}^1_{\mathcal{M}_1})|)\xrightarrow{f_1^N}H(|N(\langle\mathcal{M}_2\rangle,\mathcal{T}^2_{\mathcal{M}_2})|)\xrightarrow{f_2^N}\dots\ H(|N(\langle\mathcal{M}_n\rangle,\mathcal{T}^n_{\mathcal{M}_n})|),\tag{4}$$

where $f_i^N := H(|N(\bar{f}_i)|)$.

Persistence module (4) is not yet simplicial, but the map which sends each simplex in K to the associated linear singular simplex in $|K|$ induces an isomorphism between the simplicial homology of K and singular homology of $|K|$. Moreover, this isomorphism commutes with the maps induced in simplicial and singular homology by simplicial maps (see [29, Theorems 34.3, 34.4]). Thus, we obtain the following corollary. It facilitates the algorithmic computations of persistence diagrams for Morse decompositions of combinatorial dynamical systems.

Corollary 6.4. *The persistence diagram of* (2) *is the same as the persistence diagram of the persistence module*

$$H^\triangle(N(\langle\mathcal{M}_1\rangle,\mathcal{T}^1_{\mathcal{M}_1}))\xrightarrow{f_1^\triangle}H^\triangle(N(\langle\mathcal{M}_2\rangle,\mathcal{T}^2_{\mathcal{M}_2}))\xrightarrow{f_2^\triangle}\dots\ H^\triangle(N(\langle\mathcal{M}_n\rangle,\mathcal{T}^n_{\mathcal{M}_n})),\tag{5}$$

where $H^\triangle$ *denotes simplicial homology and* $f_i^\triangle := H^\triangle(N(\bar{f}_i))$. *Moreover, an analogous statement holds for the zigzag persistence module* (3).

For computing the persistence diagram of the module in (5), we identify the Morse sets in linear time by computing strongly connected components in G_{F_i}. The nerve of these Morse sets can also be easily computed in time linear in input mesh size (assuming the dimension of the complex to be constant). Finally, one can use the persistence algorithm in [13], specifically designed for computing the persistence diagram of simplicial maps that take the simplices of the nerve to the adjacent complexes in the sequence (4).

6.2. **Geometric interpretation.** Proposition 6.3 provides means to interpret the Alexandrov topology of subsets of simplicial complexes in the persistence module of Morse decompositions by the metric topology of their solids in the Euclidean space. Recall that the solid of a subset $A\subseteq K$ of a simplicial complex is $|A| := \bigcup\{\,\mathring{\sigma}\mid\sigma\in A\,\}$. Let simplicial complexes K_i, combinatorial dynamical systems F_i and Morse decompositions $\mathcal{M}_i$ for $i=1,2,\dots n$ be such as in Section 4. Moreover, assume $f_i : K_i\to K_{i+1}$ for $i=1,2,\dots n$ are simplicial maps. Let $\mathcal{O}^i$ denote the metric topology of the polytope $|K_i|$. Denote by $\mathcal{M}_i^{\shortparallel} := \{\,|M|\mid M\in\mathcal{M}_i\,\}$ the family of solids of Morse sets in $\mathcal{M}_i$. Consider the map $\nu_i : \langle\mathcal{M}_i^{\shortparallel}\rangle\ni x\mapsto|f_i|(x)\in\langle\mathcal{M}_{i+1}^{\shortparallel}\rangle$, which is continuous with respect to topologies $\mathcal{O}^i_{\mathcal{M}_i^{\shortparallel}}$ and $\mathcal{O}^{i+1}_{\mathcal{M}_{i+1}^{\shortparallel}}$.

Theorem 6.5. *The persistence diagram of* (2) *is the same as the persistence diagram of the persistence module*

$$H(\langle\mathcal{M}_1^{\shortparallel}\rangle,\mathcal{O}^1_{\mathcal{M}_1^{\shortparallel}})\xrightarrow{H(\nu_1)}H(\langle\mathcal{M}_2^{\shortparallel}\rangle,\mathcal{O}^2_{\mathcal{M}_2^{\shortparallel}})\xrightarrow{H(\nu_2)}\dots\xrightarrow{H(\nu_{n-1})}H(\langle\mathcal{M}_n^{\shortparallel}\rangle,\mathcal{O}^n_{\mathcal{M}_n^{\shortparallel}}).\tag{6}$$

Proof: By Proposition 6.3 it suffices to prove that the diagrams of (4) and (6) are isomorphic. By Theorem 5.1(iii), any two Morse sets in $\mathcal{M}_i$ are disconnected. Hence, it follows from Proposition 2.2 that the nerve $N(\langle\mathcal{M}_i\rangle, \mathcal{T}^i_{\mathcal{M}_i})$ splits as the disjoint union $\bigcup_{M\in\mathcal{M}_i} N(M, \mathcal{T}^i_M)$. In consequence, the whole diagram (5) splits as the direct sum of diagrams for individual Morse sets. Again by Theorem 5.1(iii), any two sets in $\mathcal{M}^{\shortparallel}_i$ are $\mathcal{O}^i_{\mathcal{M}^{\shortparallel}_i}$-disconnected. Therefore, diagram (6) also splits as the direct sum of diagrams for individual sets in $\mathcal{M}^{\shortparallel}_i$. Thus, it suffices to prove that the respective diagrams for individual Morse sets are isomorphic. This follows easily from Proposition 6.6 below. □

Since the topology of the polytope of a simplicial complex K does not depend on the choice of its geometric realization, we may assume a geometric realization of K is arbitrarily fixed. It follows from Proposition 6.1 that the barycentric subdivision of this geometric realization of K is the geometric realization of $N(K)$. Then, for any set of simplices $M \subseteq K$ we have $|N(M)| \subseteq |M|$.

Proposition 6.6. *The inclusion map $\iota_M : |N(M)| \to |M|$ is a homotopy equivalence. Moreover, if $f : K \to K'$ is a simplicial map, then the map ι_M and the map $\iota_{f(M)} : |N(f(M))| \to |f(M)|$ commute with the restrictions $|N(f)|_{||f(M)|}$ and $|f|_{||M|}$, that is $\iota_{f(M)} \circ |N(f)|_{||f(M)|} = |f|_{||M|} \circ \iota_M$.*

Proof: To prove that ι_M is a homotopy equivalence, it suffices to show that $|N(M)|$ is a deformation retract of $|M|$. To this end, order the simplices $\sigma_1, \dots, \sigma_n$ in $\operatorname{cl} M \setminus M$ so that if $\sigma_j \preceq \sigma_k$, then $k \leq j$. Let $M_0 = \operatorname{cl} M$ and $M_i = \operatorname{cl} M \setminus \{\sigma_1, \dots, \sigma_i\}$ for $i = 1, \dots, n$, and consider the sequence $\operatorname{cl} M = M_0 \supseteq M_1 \supseteq \dots \supseteq M_n = M$. We prove by induction on n that $|M|$ deformation retracts to $|N(M)|$. Observe that the poset nerve $N(M_0) = N(\operatorname{cl} M)$ coincides with the barycentric subdivision of $\operatorname{cl} M$ and thus $|M_0| = |\operatorname{cl} M| = |N(M_0)|$. See the second picture from left in Figure 4. Therefore, for $n = 0$, the claim is satisfied trivially.

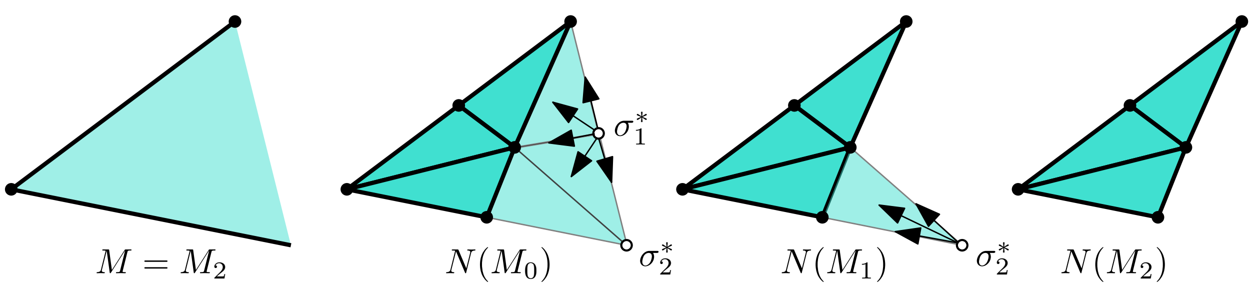

FIGURE 4. Deformation retraction of $|M|$ to $|N(M)|$. Simplices adjoining dual vertices of the absent simplices from the closure of M are shaded lighter.

Inductively assume that $|M_{i-1}|$ deformation retracts to $|N(M_{i-1})|$ for $i = 1, \dots, n$. We observe the following:

(1): In general $N(M_i) = N(M_{i-1}) \setminus C(\sigma_i)$ where $C(\sigma_i)$ denotes the set of all chains containing σ_i in the poset $(N(M_{i-1}), \preceq)$. If σ_i^* denotes the vertex corresponding to σ_i in $N(M_{i-1})$, then $C(\sigma_i)$ is the star $\operatorname{St}\sigma_i^*$ in $N(M_{i-1})$. Also, $|\sigma^*|$ is the barycenter $b(\mathring{\sigma})$.

(2): Let $Y \subseteq \operatorname{St}\sigma_i^*$ be any set of simplices in $N(M_{i-1})$ including σ_i^*. Then, $|N(M_{i-1})| \setminus |Y|$ deformation retracts to $|N(M_i)|$. This follows from the fact that $|\operatorname{St}\sigma_i^*| \setminus |\sigma_i^*| = |\operatorname{St}\sigma_i^*| \setminus b(\mathring{\sigma}_i)$ retracts to the link of σ_i^* along the segments that connect σ_i^* to the points in the link and the restriction of this retraction to points in $\operatorname{St}\sigma_i^* \setminus |Y|$ provides the necessary deformation retraction. In Figure 4, taking Y as the vertex σ_1^* along with the two edges that subdivide an absent edge in M, we see $|N(M_0)| \setminus Y$ deformation retracts to $|N(M_1)|$.

For induction, observe that $|N(M_{i-1})|$ contains a subdivision of $|\sigma_i| = \mathring{\sigma}_i$ because M_{i-1} contains σ_i and all its faces by definition of M_is. Let Y denote the set of simplices that subdivide $\mathring{\sigma}_i$. Then, according to (2), $|N(M_{i-1})| \setminus \mathring{\sigma}_i$ deformation retracts to $|N(M_i)|$; see Figure 4. We construct a deformation retraction of $|M_i|$ to $|N(M_i)|$ by first retracting $|M_{i-1}|$ to $|N(M_{i-1})|$ by the inductive hypothesis and then retracting $|N(M_{i-1})| \setminus \mathring{\sigma}_i$ to $|N(M_i)|$.

The remaining part of the lemma is an immediate consequence of Proposition 6.1. $\square$

7. Examples

In this section we present two numerical examples. The first example concerns the persistence of the Morse decompositions of a noisy sample of Kuznetsov map with respect to a frequency parameter. The second example concerns the persistence of the Morse decompositions of combinatorial multivector fields with respect to an angle parameter of the algorithm constructing the fields from a cloud of vectors.

7.1. **Kuznetsov map.** Let us consider the following planar map analyzed by Kuznetsov in the context of the Neimark-Sacker bifurcation [22, Subsection 4.6].

$$f\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) := \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}\left((1+\alpha)\begin{bmatrix}x_1\\x_2\end{bmatrix} + (x_1^2+x_2^2)\begin{bmatrix}a & -b\\ b & a\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}\right). \tag{7}$$

For parameters $\theta = \pi/17$, $\alpha = 0.5$, $a = -1$ and $b = 0.5$ the system restricted to square $[-1,1] \times [-1,1] \subseteq \mathbb{R}^2$ admits a Morse decomposition consisting of an unstable fixed point and an attracting invariant circle. (see Figure 5, upper left). We want to detect this Morse decomposition just from a finite sample of the map and in the presence of Gaussian noise. The setup is similar to the toy example in Section 3.

Let $x \in \mathbb{R}^2$ and $\epsilon_X, \epsilon_Y \in \mathbb{R}^2$ be random vectors chosen from normal distribution centered at zero, with standard deviation σ_X and σ_Y respectively. Let

$$\tilde{f}(x) := f(x + \epsilon_X) + \epsilon_Y \tag{8}$$

be a noisy version of the map (7). Consider a triangulation K of the square $Q := [-1,1] \times [-1,1] \subseteq \mathbb{R}^2$ obtained by splitting Q into a 48×48 uniform grid of squares of size $r = 1/24$ and dividing every square into two triangles. Then, the set of toplexes K_{top} consists of 4608 2-simplices. A noisy sample $\Gamma = \{(x_i, y_i)\}_i$ of the map f is generated by taking an uniformly distributed sequence of points x_i in Q and its disturbed images $y_i := \tilde{f}(x_i)$. Pairs $(x_i, y_i) \in \Gamma$

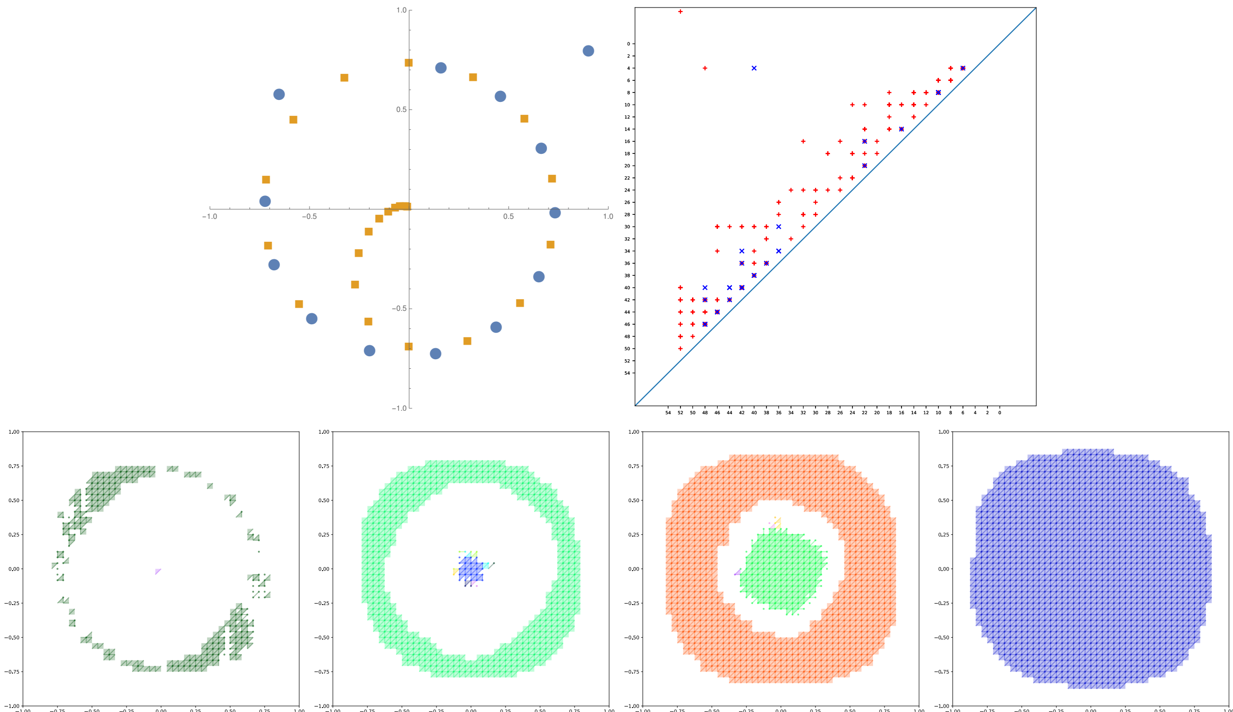

FIGURE 5. Upper left: two trajectories of (7) with starting points at $(-0.01, 0.01)$ (squares) and $(0.9, 0.8)$ (circles). Upper right: persistence diagram for a noisy sampling of (7). Red pluses and blue crosses indicate homology generators in dimension zero and one, respectively. Bottom from left to right: Morse sets (randomly colored) in selected filtration steps for threshold values $\mu = \frac{48}{148}$, $\mu = \frac{30}{148}$, $\mu = \frac{6}{148}$ and $\mu = \frac{4}{148}$, respectively.

such that $y_i \notin Q$ has been rejected from a sample. The combinatorial dynamical system F_μ is constructed in the same way as in Section 3, namely

$$F_\mu(\sigma) := \operatorname{co} \bigcup_{\tau \in K_{top},\, \sigma \preceq \tau} \{\, \bar{\tau} \in K_{top} \mid \tfrac{n_{\tau,\bar{\tau}}}{n_{max}} \geq \mu \,\}, \tag{9}$$

where $n_{\tau,\bar{\tau}}$ denotes the number of pairs in Γ connecting two toplexes $\tau, \bar{\tau} \in K_{top}$, that is

$$n_{\tau,\bar{\tau}} := \#\{(x_i, y_i) \mid x_i \in \operatorname{cl}|\tau| \text{ and } y_i \in \operatorname{cl}|\bar{\tau}|\} \tag{10}$$

and n_{max} is maximal of these values. For this particular experiment $n_{max} = 148$. Note that construction of F_μ (9) is also well defined for lower dimensional simplices.

The parameter μ in (9) describes the minimal frequency of an edge to be present in the combinatorial dynamical system F_μ. The family of Morse sets $\mathcal{M}(F_\mu)$ at given level μ consists of all strongly connected components of an associated graph. The set of considered frequency levels $1 = \mu_{74} > \mu_{73} > \ldots > \mu_0 = 0$, where $\mu_i = \frac{2i}{148}$, leads to the sequence of Morse decompositions such that $\mathcal{M}(F_{\mu_i}) \sqsubseteq \mathcal{M}(F_{\mu_{i-1}})$. The persistence diagram at Figure 5(upper right), for clarity, shows only results for $\mu \le \mu_{27} = \frac{54}{148}$, since this is the level where Morse sets start to emerge.

We experimented with various values of σ_X, σ_Y, both $\sigma_X = \sigma_Y$ and $\sigma_X \neq \sigma_Y$. The results are similar and in line with expectations as long as σ_X and σ_Y do not substantially exceed values $r/4$ and r, respectively. The detailed results for these extreme values are presented in Figure 5. The persistence diagram (Figure 5, upper right) indicates the presence of two 0-dimensional and one 1-dimensional homology generators with high persistence. Bottom row of Figure 5 shows Morse sets for some selected frequency levels. For lower thresholds, both invariant sets eventually merge together creating a unique strongly connected component. In the case without noise, the fixed point at the origin and the attracting invariant set remain separated for all values of μ.

7.2. **Lotka Volterra model.** Consider the Lotka-Volterra (LV) model:

$$\frac{\partial x}{\partial t} = x\left(1 - \frac{x}{k}\right) - \frac{(a_1 xy)}{b + x}, \qquad \frac{\partial y}{\partial t} = \frac{a_2 xy}{b + x} - gy, \tag{11}$$

where $k = 3.5$, $b = 1$, $g = 0.5$, $a_1 = (1-\frac{1}{k})(b+1)$, $a_2 = g(b+1)$ (see [6, Chapter 2, Eq. 2.13 and 2.14]). The system has a Morse decomposition consisting of a repelling stationary point and an attracting periodic orbit. We want to observe this Morse decomposition in a combinatorial dynamical system constructed from a finite sample of the vector field. In Table 1 we present an algorithm for constructing a combinatorial multivector field from a sampled vector field. The algorithm requires an angle parameter α. The constructed combinatorial multivector field and hence its combinatorial dynamical system depend on this parameter. We execute the algorithm for varying α and construct the zigzag filtration (3). Since the supporting simplicial complex (mesh) remains fixed, we obtain zigzag persistence under inclusion maps. Experiments with varying mesh, utilizing non-inclusion maps, are in progress. The outcome for the LV model is presented in Figure 6. We note that the trivial Morse sets that is Morse sets consisting of just one multivector V such that $H(\operatorname{cl} V, \operatorname{cl} V \setminus V) = 0$ are excluded from the presentation of Morse decompositions and from the barcode, because such Morse sets are considered spurious due to the triviality of their Conley index (see [27]).

The input to the algorithm CVCMF in Table 1 that computes a multivector field from a cloud of vectors consists of:

- a simplicial mesh K with vertices in a cloud of points $P = \{p_i \mid i = 1, 2, \ldots, n\} \subseteq \mathbb{R}^d$,
- the associated cloud of vectors $V := \{\vec{v_i} \mid i = 1, 2, \ldots, n\} \subseteq \mathbb{R}^d$ such that vector $\vec{v_i}$ originates from point p_i,
- an angular parameter α.

For each simplex $\sigma \in K$ and a vector $\vec{v_i}$ originating from vertex p_i of σ we measure the angle between $\vec{v_i}$ and the affine subspaces spanned by the vertices of σ. We assume the angle to be zero when the vector has length zero or the simplex is just a vertex. For a toplex σ, we assume that the angle is zero when $\vec{v_i}$ points inward σ and ∞ otherwise. When the angle is smaller than α, we project $\vec{v_i}$ onto σ. Intuitively, it aligns the vectors to the lower dimensional simplices. After this alignment, a multivector field is constructed by removing

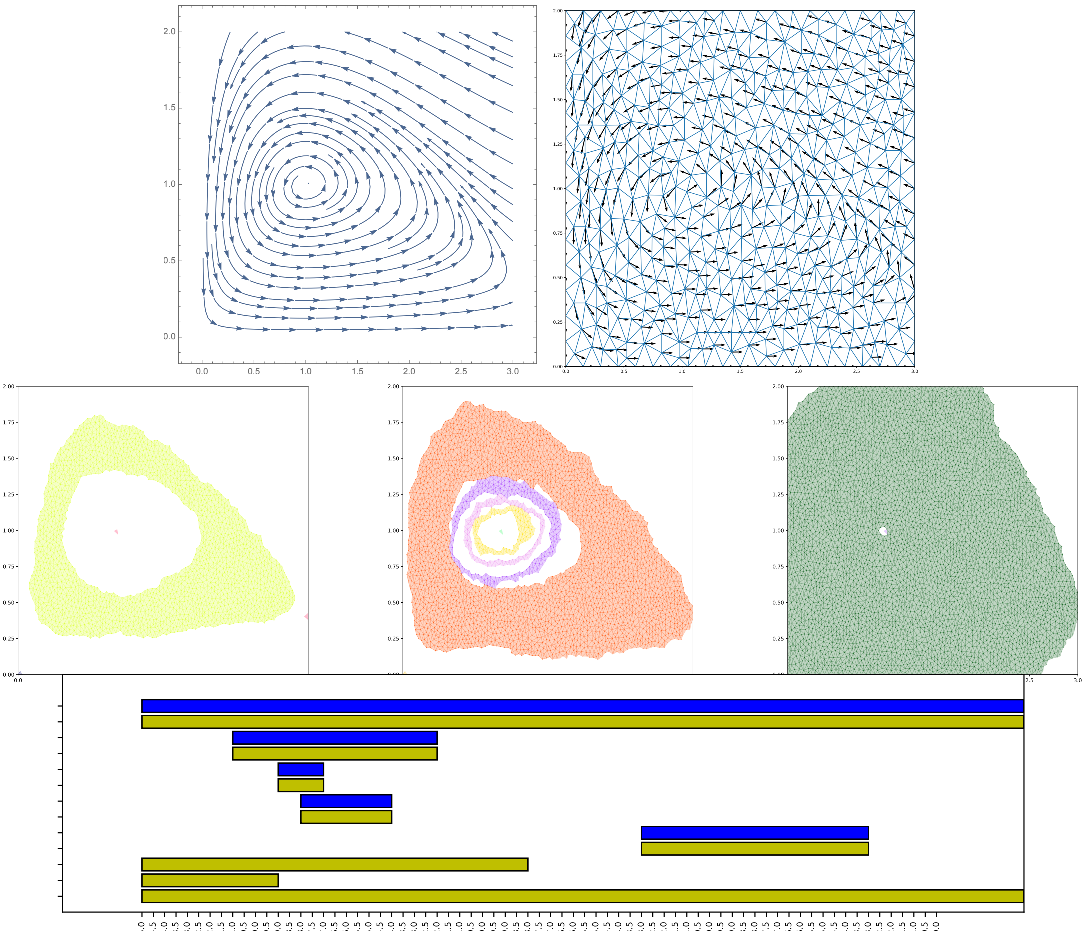

FIGURE 6. Top: A streamline plot and a cloud of vectors on a triangular grid for LV model. Middle from left to right: Morse decompositions of combinatorial dynamical systems constructed from the cloud for $\alpha = 35°$, $\alpha = 28°$ and $\alpha = 0°$, respectively. The mesh used contains 4756 points and 9300 triangles. Bottom: The respective persistence barcode. Yellow bars represents generators in dimension 0 and blue bars in dimension 1. The bars are ordered by average size of the Morse sets (biggest on the top).

the convexity conflicts. Obviously, the output depends on the parameter α. We measure changes in the multivector field $\mathcal{V}$ via persistence of its Morse decomposition. To compute such persistence we use Dionysus software [26].

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procedure CVCMF(K, V, α)
  $m \leftarrow$ an identity map $K \to K$.
  for all $\sigma \in K$ **do**
    $m[\sigma] \leftarrow$ any toplex in the star of σ pointed by mean of $\{\, \vec{v_i} \in V \mid p_i \preceq \sigma \,\}$
  for all $i = 1, 2, \ldots, n$ **do** ▷ Aligns vectors
    $S \leftarrow \{\, (\dim \sigma, \measuredangle(\sigma, \vec{v_i}), \sigma) \mid \sigma \in K$ and $p_i \preceq \sigma$ and $\measuredangle(\sigma, \vec{v_i}) \leq \alpha \,\}$
    $S' \leftarrow$ sort S using lexicographical order on first two positions ▷ $(\dim, \measuredangle, \_)$
    $(\_, \_, \sigma) \leftarrow$ first element of S'
    $m[p_i] \leftarrow \sigma$
  for all $\sigma \in K$ in descending dimension **do** ▷ Remove convexity conflicts
    while exists $\tau \preceq \sigma$ s.t. $[\tau, m[\tau]] \cap [\sigma, m[\sigma]] \neq \varnothing$ and $m[\tau] \neq m[\sigma]$ **do**
      $m[\tau] \leftarrow \sigma$ and $m[\sigma] \leftarrow \sigma$
  $\mathcal{V} \leftarrow$ build a partition of K using nonempty pre-images of m
  return $\mathcal{V}$.

TABLE 1. An algorithm constructing a combinatorial multivector field from a sampled vector field

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